

# Counting rules useful in probability, 2.4 and 2.5

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## Counting rules useful in probability, 2.4

### Theorem

*Fundamental Principle of Counting:*

*If the first task of an experiment can result in  $n_1$  possible outcomes and for each such outcome, the second task can result in  $n_2$  possible outcomes, then there are  $n_1 n_2$  possible outcomes for the two tasks together.*

# Example 1

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## ■ Solution

- *Number the balls, 1, 2, 3, 4, where ball 1 is green and balls 2, 3, 4 are blue.*
- *Total possible outcomes are  $4 \cdot 3 = 12$ .*
- *Because the balls are randomly selected, each of the 12 outcomes has a probability of  $\frac{1}{12}$ .*
- $P(A) = P(\{(1, 2)\} \cup \{(1, 3)\} \cup \{(1, 4)\} \cup \{(2, 1)\} \cup \{(3, 1)\} \cup \{(4, 1)\}) = \frac{6}{12} = \frac{1}{2}$ .

## Example 2

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Suppose you want to choose a four digit pin code for your debit card. Each digit can take the numbers 0, 1, ..., 9. How many possible choices do you have?

### ■ Solution

*Since each of the 10 digits could be chosen for each position, there are*

$$10 \times 10 \times 10 \times 10 = 10000$$

*possible choices.*

In general, if we have  $n$  objects and we select  $r$  of these with replacement, and order is important, then there are  $n^r$  possible outcomes.

## Example 3

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### ■ Solution

- *Here the selection is without replacement because the same person does not fill more than one position.*
- *Can be filled in  $7 \times 6 \times 5 = 210$  ways.*

# Permutations

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## ■ Example

The number of permutations of the set  $\{1, 2, 3\}$  is  $3! = 6$ . They are 123 132 213 231 312 321.

# Permutations

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The number of 2-permutations of the set  $\{1, 2, 3\}$  is  $\frac{3!}{(3-2)!} = 6$ .  
They are 12 21 13 31 23 32.

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*Permutations: The number of ordered arrangements or permutations  $P_k^n$  of  $k$  objects selected from  $n$  distinct objects ( $k \leq n$ ) is given by*

$$P_k^n = n(n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}.$$

In Example 3,  $P_3^7 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 7 \times 6 \times 5 = 210$ .



# Combinations

## Definition

A combination of a set of objects is a subset of those objects.

## Theorem

*Combinations: The number of distinct subsets or combinations of size  $k$  that can be selected from  $n$  distinct objects ( $k \leq n$ ) is given by*

$$C_k^n = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

*This is the same as the number of ways of partitioning  $n$  distinct objects into exactly two subsets containing  $k$  and  $(n - k)$  elements, respectively.*

# Combinations

In R, the command for  $\binom{n}{k}$  is: `choose(n,k)`

## Example

There are  $\binom{3}{2} = \frac{3!}{2!(3-2)!} = 3$  subsets of size 2 of the set  $\{1, 2, 3\}$ .  
They are  $\{1, 2\}$ ,  $\{1, 3\}$ ,  $\{2, 3\}$ .

- If order matters, use permutations.
- If order does not matter, use combinations.

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- $\binom{n}{n}$  subsets with  $n$  elements.
- The total number of possible subsets of  $S$  is  $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$ .

# Binomial formula

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$$(1 + a)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^k$$

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- Hence, if a set has  $n$  elements, it has  $2^n$  different subsets.

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Find the first three terms in the binomial expansion of  $(1 + x)^{\frac{1}{3}}$ .

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## ■ Solution

$$\binom{\frac{1}{3}}{2} = \frac{\frac{1}{3}(\frac{1}{3} - 1)}{2!} = \frac{\frac{1}{3}(-\frac{2}{3})}{2} = -\frac{1}{9}.$$

So

$$(1 + x)^{\frac{1}{3}} \approx 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \dots$$

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## ■ Solution

*The number of possible winning numbers is the combination of 6 objects selected from 48, that is,*

$$\binom{48}{6} = \frac{48!}{6!(48-6)!} = 12271512.$$



# Example

Since each of the 12271512 possible winning numbers are equally likely, the probability that a player who buys one ticket wins is  $\frac{1}{12271512}$ .

## Example

A box contains 15 balls; 3 red, 5 blue and 7 green balls. 8 balls are selected from the box without replacement. What is the probability that the balls selected are 1 red , 4 blue and 3 green?

# Solution

- The number of ways to select 8 balls from 15 is,  
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$$\binom{5}{4} = \frac{5!}{4!(5-4)!} = 5.$$

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- The number of ways to select 3 green balls from 7 green is,  
$$\binom{7}{3} = \frac{7!}{3!(7-3)!} = 35.$$

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- The number of ways to select 3 green balls from 7 green is,  
$$\binom{7}{3} = \frac{7!}{3!(7-3)!} = 35.$$
- Then by the Fundamental Principle of Counting, the number of ways to select 1 red ball, 4 blue balls and 3 green is,  
$$\binom{3}{1} \binom{5}{4} \binom{7}{3} = 3 \cdot 5 \cdot 35 = 525.$$

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$$\binom{3}{1} \binom{5}{4} \binom{7}{3} = 3 \cdot 5 \cdot 35 = 525.$$
- The probability of selecting 1 red ball, 4 blue balls and 3 green balls is,  
$$\frac{\binom{3}{1} \binom{5}{4} \binom{7}{3}}{\binom{15}{8}} = \frac{525}{6435} = \frac{105}{1287}.$$

# Example

## Example

A hand of seven cards is dealt from a deck of 52 cards.

(A) How many hands are possible?

(B) How many hands that contain exactly 2 diamonds are possible?

(C) How many hands with 2 diamonds and 3 spades are possible?

# Solution

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- (A) *The number of seven-card hands is the number of choosing 7 elements from a set of 52 which is  $\binom{52}{7}$ .*

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- (A) *The number of seven-card hands is the number of choosing 7 elements from a set of 52 which is  $\binom{52}{7}$ .*
- (B) *A hand with 2 diamonds can be formed as follows. Imagine,*
  - *deal 2 diamonds from the 13 diamonds.*
  - *deal 5 cards from the remaining 39 cards.*

*The number of possible hands is,*

$$\binom{13}{2} \binom{39}{5} = 44909046.$$

# Solution

## Solution

(C) A hand with 2 diamonds and 3 spades can be formed as follows.

- deal 2 diamonds from the 13 diamonds.
- deal 3 spades from the 13 spades.
- deal 2 cards from the remaining 26 cards.

The number of possible hands is,

$$\binom{13}{2} \binom{13}{3} \binom{26}{2} = 7250100.$$

## Theorem

*Partition: The number of ways of partitioning  $n$  distinct objects into  $k$  groups containing  $n_1, n_2, \dots, n_k$  objects, respectively, is*

$$\frac{n!}{n_1!n_2! \cdots n_k!},$$

where

$$\sum_{i=1}^k n_i = n.$$

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In how many ways can we permute the letters of the word MISSISSIPPI?

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## ■ Solution

$$n = 11$$

$$n_1 = 1 \text{ M}$$

$$n_2 = 4 \text{ I's}$$

$$n_3 = 4 \text{ S's}$$

$$n_4 = 2 \text{ P's}$$

Hence, number of permutations is

$$\frac{11!}{1!4!4!2!} = 34650.$$



# Example

## ■ Example

A coin is flipped 5 times, and the number of heads and tails is counted.

(A) How many flips have exactly 2 heads?

(B) How many flips have at most 2 heads?

(C) How many flips have at least 2 heads?

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## Solution

*(A) If there are exactly 2 heads, there are exactly 3 tails, so the number of such flips is the number of permutations of 2 heads and 3 tails. That number is  $\frac{5!}{2!3!} = 10$ .*

- (B) At most 2 heads means that one of the following is true:
  - no heads
  - exactly 1 head
  - exactly 2 heads
  - Hence  $\frac{5!}{0!5!} + \frac{5!}{1!4!} + \frac{5!}{2!3!} = 1 + 5 + 10 = 16$  flips have at most 2 heads.

- (B) At most 2 heads means that one of the following is true:
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  - Hence  $\frac{5!}{0!5!} + \frac{5!}{1!4!} + \frac{5!}{2!3!} = 1 + 5 + 10 = 16$  flips have at most 2 heads.
- (C) At least 2 heads means that one of the following is true:
  - exactly 2 heads
  - exactly 3 heads
  - exactly 4 heads
  - exactly 5 heads
  - Hence  $\frac{5!}{2!3!} + \frac{5!}{3!2!} + \frac{5!}{4!1!} + \frac{5!}{5!0!} = 10 + 10 + 5 + 1 = 26$  flips have at least 2 heads.

# Example

## Example

A coin is flipped  $n$  times.

- Total number of outcomes is the number of sequences of length  $n$  which is  $2^n$ .
- The number of sequences with  $k$  heads, is the number of subsets of size  $k$  of a set with  $n$  elements, which is  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ .
- The probability of  $k$  heads in a sequence of  $n$  coin tosses is

$$\frac{\binom{n}{k}}{2^n}.$$

## More counting rules useful in probability, chapter 2.5

### Theorem

*The number of ways of making  $k$  selections from  $n$  objects when selection is made with replacement and order is not important is*

$$\binom{n + k - 1}{k}$$

# Example

## Example

Suppose in a lottery, a player selects 6 numbers from 1 to 48. During the draw, 6 numbers are drawn at random and with replacement. To win the first prize, all 6 numbers must match those drawn in any order. How many winning numbers are possible?

## Solution

*The number of possible winning numbers is*

$$\binom{48 + 6 - 1}{6} = \frac{53 \times 52 \times 51 \times 50 \times 49 \times 48}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 22957480.$$

# Summary

	Order is Important	Order is Not Important
With Replacement	$n^k$	$\binom{n+k-1}{k}$
Without Replacement	$P_k^n = \frac{n!}{(n-k)!}$	$C_k^n = \binom{n}{k}$

# Birthday problem, chapter 2.4

## Example

Suppose  $k$  people are in a room

(A) what is the probability that no two of them have the same birthday?

(B) How many people must be in the room for the probability that at least two of the  $k$  people have the same birthday to be greater than  $\frac{1}{2}$ .

# Birthday problem, Solution

## Solution

(A) Since there are  $k$  people in the room and each has 365 possibilities for a birthday, there are  $365^k$  possible outcomes. The number of possible outcomes for which no 2 have the same birthday is  $(365)(364)\dots(365 - k + 1)$ . Thus, the probability, under the assumption that each outcome is equally likely, that no 2 people in the room have the same birthday is,  $\frac{(365)(364)\dots(365 - k + 1)}{365^k}$ .

(B) The probability that at least two people of the  $k$  people share a birthday is  $1 - \frac{(365)(364)\dots(365 - k + 1)}{365^k}$ . Note that we can write  $\frac{(365)(364)\dots(365 - k + 1)}{365^k} = \frac{(365)(364)\dots(365 - (k - 1) + 1)}{365^{k-1}} \frac{(365 - k + 1)}{365}$ . Using  $R$  (see other slide), we obtained that when  $k = 23$ , there is a 50% chance that at least two of the people in the room share a birthday. In a class of 38 students, there is a 86% chance that at least two people share a birthday.



## Example

Suppose a box contains 30 balls, of which 12 is white and 18 is black. Pick 4 balls without replacement. The total possible outcomes are  $\binom{30}{4}$ . Let  $k$  be the number of white balls chosen. Then we have chosen  $k$  white balls out of 12 and  $4 - k$  black balls out of 18. Then by the Fundamental Theorem of Counting, there are  $\binom{12}{k} \cdot \binom{18}{4-k}$  possible outcomes. Thus, the probability of choosing  $k$  white balls is then

$$\frac{\binom{12}{k} \cdot \binom{18}{4-k}}{\binom{30}{4}}.$$

## Example

```
> k <- -c(0 : 4)
> P <- -(choose(12, k) * choose(18, 4 - k))/(choose(30, 4))
> data.frame(k, P)
  k  P
1  0 0.11165846
2  1 0.35730706
3  2 0.36847291
4  3 0.14449918
5  4 0.01806240
```